

# The Electrical Conductivity of a Partially Ionized Argon-Potassium Plasma in a Magnetic Field

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In the case of small deviations from thermal equilibrium the second ENSKOG approximation is used as a starting point for solving the BOLTZMANN equation of the electrons in a partially ionized plasma. The distribution function is expanded according to LAGUERRE polynomials up to the order of 3. In this order the electrical conductivity of a LORENTZ gas, which is known exactly, is obtained to an accuracy of roughly 5%. The approximation tested in this way was then used to calculate the conductivity of an argon-potassium mixture at electron temperatures between 2000 °K and 3500 °K.

If only the collisions between electrons and argon atoms were to be considered, the electrical conductivity in the absence of a magnetic field would, in view of the RAMSAUER effect, be greater by a factor of 2.8 than that obtained with an infinitely strong magnetic field. When the interaction with the potassium atoms and the COULOMB interaction are taken into account as well the conductivity in the magnetic field varies by about 20%.

In the theory of the electrical conductivity of a partially ionized plasma there are two well-known limiting cases. For the case where the ionization is so weak that the electron-ion collisions can be ignored in relation to the collisions between electrons and neutral particles a complete theory of mobility of electrons in electric and magnetic fields has been developed by ALLIS<sup>1</sup>. The other limiting case of the fully ionized plasma is exactly known in the same way<sup>2</sup>.

The object of this paper is to use the second ENSKOG approximation<sup>3</sup> to obtain an approximate solution of the BOLTZMANN equation that describes the entire transition from a weak to a fully ionized plasma in a magnetic field, as an extension of the calculations without magnetic field already made<sup>4</sup>. That is, the expansion of the distribution function  $f_1(\mathbf{u}, \mathbf{r}, t)$  of the electrons

$$f_1 = f_1^{(0)} + f_1^{(1)} + f_2^{(2)} + \dots$$

is interrupted at  $f_1^{(1)}$ , where  $\mathbf{u}$  is the velocity of a particle in the laboratory system. The local MAXWELL distribution appears as the function of lowest order  $f_1^{(0)}$ . The equation for the second approximation

$$f_1^{(1)} = f_1^{(0)} \Phi_1, \quad |\Phi_1| \ll 1$$

depends linearly on those variables which cause a perturbation of the thermal equilibrium. These are an electric field, a temperature gradient and a pres-

sure gradient. The higher orders  $f_1^{(2)}$ ,  $f_1^{(3)}$  ... etc., which are neglected, contain none of these linear terms.

This paper is only concerned with the influence exerted on the distribution function by the electric field determining the expression for OHM's law. Thus, the coefficient of the thermal diffusion also contained therein shall not be treated either.

To enable the MAXWELL distribution to be chosen as zero-order distribution function the electric field has to be so weak that the drift velocity between electrons and heavy particles is small with respect to the thermal velocity. It is permissible here for the electron temperature, because of the OHMIC heating, to differ from the temperature of the heavy particles, i. e. the local MAXWELL distribution should be based on the mean electron temperature. The electron density should therefore be sufficiently high so that the energy exchange between the electrons takes place faster than the energy exchange with the heavy particles.

## The Second Enskog Approximation

In order to obtain a solution for the distribution function the following assumption is made:

$$f_1 = f_1^{(0)} \left\{ 1 - \frac{e\tau}{kT_1} \sqrt{\frac{kT_1}{m}} \left[ (\mathbf{E}_{\parallel}^*, \boldsymbol{\zeta}) \chi_1 + (\mathbf{E}_{\perp}^*, \boldsymbol{\zeta}) \chi_2 - \left( \left[ \mathbf{E}_{\perp}^* \times \frac{\mathbf{B}}{B} \right], \boldsymbol{\zeta} \right) \chi_3 \frac{2\omega\tau}{3\sqrt{2}\pi} \right] \right\} \quad (1)$$

<sup>1</sup> W. P. ALLIS, Motions of Ions and Electrons, Handbuch der Physik, Band XXI, Berlin 1956.

<sup>2</sup> R. LANDSHOFF, Phys. Rev. 76, 904 [1949].

<sup>3</sup> S. CHAPMAN and T. G. COWLING, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, London and New York 1939.

<sup>4</sup> R. S. DE VOTO, Argon Plasma Transport Properties, Report No. SU AA217, Stanford University, Stanford, California.



where  $T_1$  is the electron temperature,  $m$  the electron mass,  $e$  the electron charge,  $\mathbf{B}$  the magnetic field,

$$\mathbf{E}^* = \mathbf{E} + \frac{1}{c} [\mathbf{v} \times \mathbf{B}]$$

the generalized electric field,  $\mathbf{v}$  the centre of mass velocity,

$$\boldsymbol{\zeta} = (\mathbf{u} - \mathbf{v}) \sqrt{k T_1 / m}$$

a dimensionless random velocity,  $\tau = 1/\nu$  the collision time,  $\nu = \sum_{\alpha} \nu_{1\alpha}$  the collision frequency for momentum exchange between electrons and the other gas components such as ions ( $\alpha=2$ ), argon atoms ( $\alpha=3$ ) and potassium atoms ( $\alpha=4$ ). These collision frequencies are defined by

$$\nu_{12} = \frac{4\sqrt{2}\pi \ln \Lambda e^4 n_2}{3\sqrt{m}(kT)^{3/2}}, \quad (2)$$

$$\nu_{1\beta} = \frac{2\sqrt{kT_1}}{3\sqrt{2}\pi m} n_{\beta} \int_0^{\infty} \zeta^5 Q_m^{(\beta)} e^{-\zeta^2/2} d\zeta, \quad \zeta = |\boldsymbol{\zeta}|$$

where  $\ln \Lambda$  is the COULOMB logarithm,  $n_{\alpha}$  the particle densities. The momentum exchange cross section  $Q_m^{(\beta)}$  arises from the collisions between electrons and neutral atoms ( $\beta=3, 4$ ). This generally depends on the energy of the colliding electron. The electric field  $\mathbf{E}^* = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}$  is divided into two components parallel and perpendicular to the magnetic field.

This assumption for  $f_1$  is substituted in the second ENSKOG approximation of the BOLTZMANN equation. The COULOMB interaction was described by means of the FOKKER-PLANCK equation in the form given by LANDAU. The collision integrals for the collisions between electrons and heavy particles were used in the approximation of an infinite mass ratio. For the functions  $\chi_1, \chi_2, \chi_3$ , which depend only on the value of the particle velocity  $\zeta$ , the following equations are obtained

$$\frac{2}{3\sqrt{2}\pi} = \frac{\nu_{12}}{\nu} \left( \frac{\chi_1}{\zeta^3} + \frac{1}{3\sqrt{2}\pi} F(\chi_1) \right) \quad (3)$$

$$+ \frac{2}{3\sqrt{2}\pi} \chi_1 \left[ \frac{\nu_{13}}{\nu} + \frac{\nu_{14}}{\nu} \right],$$

$$\frac{2}{3\sqrt{2}\pi} - \left( \frac{2\omega\tau}{3\sqrt{2}\pi} \right)^2 \chi_3 = \frac{\nu_{12}}{\nu} \left( \frac{\chi_2}{\zeta^3} + \frac{1}{3\sqrt{2}\pi} F(\chi_2) \right) + \frac{2}{3\sqrt{2}\pi} \chi_2 \left[ \frac{\nu_{13}}{\nu} + \frac{\nu_{14}}{\nu} \right], \quad (4)$$

$$\chi_2 = \frac{\nu_{12}}{\nu} \left( \frac{\chi_3}{\zeta^3} + \frac{1}{3\sqrt{2}\pi} F(\chi_3) \right) + \frac{2}{3\sqrt{2}\pi} \chi_3 \left[ \frac{\nu_{13}}{\nu} + \frac{\nu_{14}}{\nu} \right],$$

$$\bar{\nu}_{13} = n_3 \sqrt{\frac{kT_1}{m}} \zeta Q_m^{(3)}(\zeta), \quad \bar{\nu}_{14} = n_4 \sqrt{\frac{kT_1}{m}} \zeta Q_m^{(4)}(\zeta).$$

The first set of brackets on the right-hand side of (3), (4) gives the values of the electron-ion and electron-electron collisions. The latter are allowed for by a complicated expression  $F(\chi)$  taken from a paper on the transport coefficients of a fully ionized plasma<sup>5</sup>:

$$\begin{aligned} F(\chi) = & \chi \left[ \frac{12}{\zeta^3} \int_0^{\zeta} x^2 e^{-x^2/2} dx - 6 e^{-\zeta^2/2} \right] \\ & - \frac{d\chi}{d\zeta} \left[ \left( \frac{6}{\zeta^4} - \frac{6}{\zeta^2} \right) \int_0^{\zeta} x^2 e^{-x^2/2} dx + \frac{6}{\zeta} e^{-\zeta^2/2} \right] \\ & - \frac{d^2\chi}{d\zeta^2} \frac{6}{\zeta^3} \int_0^{\zeta} x^2 e^{-x^2/2} dx \\ & + \frac{2}{\zeta^3} \int_0^{\zeta} x^4 \chi(x) e^{-x^2/2} dx \\ & - \frac{6}{5\zeta^3} \int_0^{\zeta} x^6 \chi(x) e^{-x^2/2} dx \\ & + \left( 2 - \frac{6}{5} \zeta^2 \right) \int_0^{\infty} x \chi(x) e^{-x^2/2} dx. \end{aligned} \quad (5)$$

### The Electrical Conductivity

OHM's law is obtained as the first-order moment of the distribution function (1):

$$\mathbf{i}_{\perp} = \sigma_{\perp} \mathbf{E}_{\perp} - (\omega\tau)_{\text{eff}} [\mathbf{i} \times \mathbf{B}/B] \quad (6)$$

where  $\mathbf{i}_{\perp}$  is the current density perpendicular to the magnetic field.

The electrical conductivity  $\sigma_{\perp}$  and the HALL parameter  $(\omega\tau)_{\text{eff}}$  are connected with the functions  $\chi_2$  and  $\chi_3$ :

$$\sigma_{\perp} = \alpha_1 (1 + \alpha_2^2/\alpha_1^2), \quad (\omega\tau)_{\text{eff}} = \alpha_2/\alpha_1, \quad (7)$$

$$\alpha_1 = \frac{n_1 e^2}{m\nu} \frac{2}{3\sqrt{2}\pi} \int_0^{\infty} \zeta^4 \chi_2 e^{-\zeta^2/2} d\zeta,$$

$$\alpha_2 = \frac{n_1 e^2}{m\nu} \left( \frac{2}{3\sqrt{2}\pi} \right) \frac{2\omega\tau}{3\sqrt{2}\pi} \int_0^{\infty} \zeta^4 \chi_3 e^{-\zeta^2/2} d\zeta.$$

For a strong magnetic field  $\omega\tau \rightarrow \infty$  it can easily be seen that the Eqs. (4) have the solution:

$$\begin{aligned} \chi_3 &= 3\sqrt{2}\pi/2 (\omega\tau)^2, \\ \chi_2 &= \frac{1}{(\omega\tau)^2} \left\{ \frac{3\sqrt{2}\pi}{2\nu\zeta^3} \nu_{12} + \frac{\nu}{\nu_{13}} + \frac{\bar{\nu}_{14}}{\nu} \right\}. \end{aligned} \quad (8)$$

<sup>5</sup> W. FENEBERG, Transporterscheinungen eines vollionisierten Wasserstoffplasmas im Magnetfeld, Dissertation Universität München, Januar 1966.

From this it follows that in the limiting case of an infinitely strong magnetic field the simple theory again holds:

$$\omega \tau \rightarrow \infty, \quad \sigma_{B=\infty} = n_1 e^2 / m \nu, \quad (\omega \tau)_{\text{eff}} = \omega / \nu. \quad (9)$$

Generally, however, both the conductivity and the HALL parameter are always greater than is described by the simple theory.

For calculating an approximate solution of Eq. (4) the functions  $\chi_2$  and  $\chi_3$  were developed according to LAGUERRE polynomials

$$\chi_2 = \sum_{p=0}^l \alpha_p L_p^{(3/2)}(\zeta^2/2), \quad \chi_3 = \sum_{p=0}^l b_p L_p^{(3/2)}(\zeta^2/2)$$

and the series was stopped at the order  $l=3$ . This is a sufficiently close approximation of the exact results of the LORENTZ gas model obtained simply by leaving out  $F(\chi)$ .

Table 1 shows for the LORENTZ gas the ratio  $\delta = \sigma_{B=0} / \sigma_{B=\infty}$  of the electrical conductivity without a magnetic field  $\sigma_{B=0}$  to the electrical conductivity with an infinitely strong magnetic field  $\sigma_{B=\infty}$ . The difference between the exact solution and the ap-

$T_1$ °K	$\delta = \delta_{B=0} / \delta_{B=\infty}$	
	Exact	$l=3$
2000	1.19	1.14
2200	1.08	1.08
2400	1.21	1.15
2600	1.25	1.17
2800	1.32	1.26
3000	1.40	1.35
3200	1.49	1.46
3400	1.58	1.50

Table 1. LORENTZ Gas Model.

proximation used  $l=3$  is only about 5%. It is thus permissible to use this approximation for the real

gas as well. A significant part is played by the RAMSAUER effect with regard to the collisions between electrons and argon atoms. The dependence of the cross section on the energy of the colliding electron is already well known from the paper by FROST and PHELPS<sup>6</sup>, which was used here. The cross section of the electron-potassium collisions is unfortunately not at all so well known. A constant cross section  $Q = 4 \times 10^{-14} \text{ cm}^2$  (l. c.<sup>7</sup>) was used here for the calculations.

The degree of ionization present at a given electron temperature was calculated with the SAHA equation from the ionization of potassium. Table 2 shows

$T_1$ °K	$\delta_{\perp} [\Omega^{-1} \text{ cm}^{-1}]$			$(\omega \tau)$ $B=10 \text{ kG}$	$(\omega \tau)_{\text{eff}} / \omega \tau$ $B=10 \text{ kG}$
	$B=0$	$B=10 \text{ kG}$	$B=\infty$		
2000	0.081	0.072	0.072	10.0	1.01
2200	0.237	0.224	0.223	9.31	1.00
2400	0.498	0.444	0.438	6.66	1.02
2600	0.891	0.810	0.786	5.08	1.04
2800	1.44	1.30	1.22	3.79	1.09
3000	2.09	1.93	1.69	2.83	1.18
3200	2.83	2.66	2.18	2.15	1.27
3400	3.53	3.39	2.67	1.69	1.33

Table 2. Electrical Conductivity and HALL Parameter for an Argon-Potassium Mixture.  $T_g=2000^\circ\text{K}$ , Potassium: 0.2%.

for various electron temperatures the dependence of the values  $\sigma_{\perp}$  and  $(\omega \tau)_{\text{eff}}$  on the magnetic field in the case of a potassium concentration of 0.2%. A gas temperature of  $2000^\circ\text{K}$  and a pressure of 1 atm. were assumed.

The corresponding measurements were made by BREDERLOW and HODGSON<sup>8</sup>.

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<sup>6</sup> L. S. FROST and A. V. PHELPS, Phys. Rev. **136** A 1538 [1964].

<sup>7</sup> R. B. BRODE, Phys. Rev. **34**, 673 [1929].

<sup>8</sup> G. BREDERLOW, W. FENEBERG, and R. HODGSON, Non-Equilibrium Conductivity in an Argon Potassium Plasma in Crossed Electric and Magnetic Fields, International Symposium on Magnetohydrodynamic Electrical Power Generation, Salzburg 1966, Paper No. SM-74/15.